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Cutting corners with spheres in d-dimensions

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Abstract

A sphere corner cut $\mathbf{A} \subset \mathbb{N}_0^d$ is a set of points with nonnegative integer coordinates which includes the origin and which can be separated from $\mathbb{N}_0^d \setminus \mathbf{A}$ by a d-dimensional sphere. We show that in a given d-dimensional space there are $O(n^{d+1}(\log n)^{d-1})$ sphere corner cuts consisting of n points.

Keywords: Corner cuts; Partitions; Integer grid

1. Introduction

A sphere corner cut $\mathbf{A} \subset \mathbb{N}_0^d$ is a set of points with nonnegative integer coordinates which includes the origin $\mathbf{0}$ and which can be separated from $\mathbb{N}_0^d \setminus \mathbf{A}$ by a sphere. In this paper we show that for a fixed d there are $O(n^{d+1}(\log n)^{d-1})$ sphere corner cuts with cardinality n.

Problems related to the cutting corners by hyperplanes in d-dimensional spaces are considered recently in [2,7,8]. If our method is applied it gives an $O(n^{d-1/d}(\log n)^{d-1})$ upper bound for the number of hyperplane corner cuts. The last upper bound is not so good as the bound $O(n^{d-1}(\log n)^{d-1})$ recently derived in [8] but (for $d > 2$) it is better than $O(n^{(2d(d-1))/(d+1)})$-bound derived in [7].

The paper is organized as follows. At the end of this section we give the basic definitions and denotations. Section 2 presents an efficient corner cuts characterization. The main result is derived in Section 3, where the number of sphere corner cuts with n points is

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estimated. In Section 4, the same method is applied to enumerating hyperplane corner cuts. Section 5 gives concluding remarks.

Throughout this paper, $\mathbb{N}_0$ denotes the set $\{0, 1, 2, \ldots\}$, $\mathbb{R}$ denotes the set of real numbers, and $\#\mathbf{A}$ is the cardinality of the set $\mathbf{A}$.

Definition 1. A set $\mathbf{A} \subset \mathbb{N}_0^d$ is a sphere corner cut if $\mathbf{0} \in \mathbf{A}$ and if there are real numbers $a_1, a_2, \ldots, a_d$, and r such that

$$\sum_{i=1}^{d}(x_i - a_i)^2 < r^2 \quad \text{for } \mathbf{x} \in \mathbf{A} \quad \text{and} \quad \sum_{i=1}^{d}(x_i - a_i)^2 > r^2 \quad \text{for } \mathbf{x} \in \mathbb{N}_0^d \setminus \mathbf{A}.$$

A sphere corner cut $\mathbf{A}$ will be called *n-point sphere corner cut* if $\#\mathbf{A} = n$.

For a finite subset $\mathbf{A} \subset \mathbb{R}^d$ we define:

$$\mu_i(\mathbf{A}) := \sum_{\mathbf{x} \in \mathbf{A}} x_i, \quad i = 1, \ldots, d.$$

2. Sphere corner cuts characterization

The next theorem shows that the correspondence $\mathbf{A} \to (\mu_1(\mathbf{A}), \ldots, \mu_d(\mathbf{A}))$ is one-to-one if $\mathbf{A}$ is an n-point sphere corner cut.

Theorem 2. *If $\mathbf{A}$ and $\mathbf{B}$ are n-point sphere corner cuts then $\mu_i(\mathbf{A}) = \mu_i(\mathbf{B})$ for all $1 \leqslant i \leqslant d$ implies $\mathbf{A} = \mathbf{B}$.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be given. From the definition, there exists a sphere $\mathcal{S}_1 = \{\mathbf{x}\colon (x_1 - a_1)^2 + \cdots + (x_d - a_d)^2 = r_1^2\}$ which separates $\mathbf{A}$ and $\mathbb{N}_0^d \setminus \mathbf{A}$ and another sphere $\mathcal{S}_2 = \{\mathbf{x}\colon (x_1 - b_1)^2 + \cdots + (x_d - b_d)^2 = r_2^2\}$ which separates $\mathbf{B}$ and $\mathbb{N}_0^d \setminus \mathbf{B}$. We will prove that $\mu_i(\mathbf{A}) = \mu_i(\mathbf{B})$ $(1 \leqslant i \leqslant d)$ and $\mathbf{A} \neq \mathbf{B}$ leads to a contradiction.

Obviously, if

$$\mathcal{L} = \{\mathbf{x}\colon c_1 x_1 + \cdots + c_d x_d + h = 0\}$$

is the hyperplane determined by the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ (specifically $c_i = 2 \cdot (b_i - a_i)$ for $1 \leqslant i \leqslant d$ and $h = r_2^2 - r_1^2 + \sum_{i=1}^{d}(a_i^2 - b_i^2)$) then $\mathcal{L}$ separates $\mathbf{A} \setminus \mathbf{B}$ and $\mathbf{B} \setminus \mathbf{A}$. We may assume that

$$c_1 x_1 + \cdots + c_d x_d + h \begin{cases} > 0 & \text{for } \mathbf{x} \in \mathbf{A} \setminus \mathbf{B}, \\ < 0 & \text{for } \mathbf{x} \in \mathbf{B} \setminus \mathbf{A}. \end{cases} \tag{1}$$

Then

$$0 < \sum_{i=1}^{d} c_i \mu_i (\mathbf{A} \setminus \mathbf{B}) - \sum_{i=1}^{d} c_i \mu_i (\mathbf{B} \setminus \mathbf{A})$$
$$= \left(\sum_{i=1}^{d} c_i \mu_i (\mathbf{A}) - \sum_{i=1}^{d} c_i \mu_i (\mathbf{B} \cap \mathbf{A}) \right) - \left(\sum_{i=1}^{d} c_i \mu_i (\mathbf{B}) - \sum_{i=1}^{d} c_i \mu_i (\mathbf{B} \cap \mathbf{A}) \right) = 0,$$

a contradiction. □

3. Estimating the number of sphere corner cuts

In this section we derive an upper bound for the number of n-point sphere corner cuts. To do so, we need an auxiliary lemma.

Lemma 3. *Let d be fixed and $\mathbf{A}$ be an n-point sphere corner cut. Then*

$$(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_d) \in \mathbf{A} \quad \Rightarrow \quad \prod_{i=1}^{d} \bar{x}_i = O(n), \tag{2}$$

$$\left\{(\bar{x}_{i,1}, \bar{x}_{i,2}, \ldots, \bar{x}_{i,d}):\ i = 1, \ldots, d\right\} \subset \mathbf{A} \quad \Rightarrow \quad \prod_{i=1}^{d} \bar{x}_{i,i} = O(n). \tag{3}$$

Proof. Let the sphere $\{\mathbf{x}:\ (x_1 - a_1)^2 + \cdots + (x_d - a_d)^2 = r^2\}$ separate $\mathbf{A}$ and $\mathbb{N}_0^d \setminus \mathbf{A}$. Then $\bar{\mathbf{x}} \in \mathbf{A}$ and $\mathbf{0} \in \mathbf{A}$ imply

$$(\bar{x}_1 - a_1)^2 + \cdots + (\bar{x}_d - a_d)^2 < r^2 \quad \text{and} \quad a_1^2 + \cdots + a_d^2 < r^2. \tag{4}$$

From (4) we have

$$\left((\bar{x}_1 - a_1)^2 + a_1^2\right) + \cdots + \left((\bar{x}_d - a_d)^2 + a_d^2\right) < 2 \cdot r^2,$$

which leads to the conclusion:

- either $\bar{\mathbf{x}}_1' = (\bar{x}_1, 0, \ldots, 0) \in \mathbf{A}$ or $\bar{\mathbf{x}}_1'' = (0, \bar{x}_2, \ldots, \bar{x}_d) \in \mathbf{A}$;
- either $\bar{\mathbf{x}}_2' = (0, \bar{x}_2, \ldots, 0) \in \mathbf{A}$ or $\bar{\mathbf{x}}_2'' = (\bar{x}_1, 0, \ldots, \bar{x}_d) \in \mathbf{A}$;
- ...
- either $\bar{\mathbf{x}}_d' = (0, 0, \ldots, \bar{x}_d) \in \mathbf{A}$ or $\bar{\mathbf{x}}_d'' = (\bar{x}_1, \bar{x}_2, \ldots, 0) \in \mathbf{A}$.

At least one of points $\bar{\mathbf{x}}_i'$, $1 \leqslant i \leqslant d$, belongs to $\mathbf{A}$ since

$$\bar{\mathbf{x}}_i' \notin \mathbf{A} \quad \text{for } i = 1, \ldots, d$$
$$\Rightarrow \quad (\bar{x}_1 - a_1)^2 + \cdots + (\bar{x}_d - a_d)^2 + (d-1) \cdot \left(a_1^2 + \cdots + a_d^2\right) > d \cdot r^2$$

contradicts (4). Similarly, at least one of points $\bar{\mathbf{x}}_i''$ $(1 \leqslant i \leqslant d)$ belongs to **A**, otherwise

$$a_1^2 + \cdots + a_d^2 + (d-1) \cdot \big((\bar{x}_1 - a_1)^2 + \cdots + (\bar{x}_d - a_d)^2\big) > d \cdot r^2,$$

contradicting (4).

Hence, we can conclude:

- if $d = 2$ then either $\{\mathbf{0}, (\bar{x}_1, 0), (\bar{x}_1, \bar{x}_2)\} \subset \mathbf{A}$ or $\{\mathbf{0}, (0, \bar{x}_2), (\bar{x}_1, \bar{x}_2)\} \subset \mathbf{A}$;
- if $d > 2$ then, up to a permutation of coordinates $x_1, \ldots, x_d$, there is an index k, with $1 < k < d$, such that

$$\big\{\mathbf{0}, \bar{\mathbf{x}}_1', \ldots, \bar{\mathbf{x}}_k', \bar{\mathbf{x}}_{k+1}'', \ldots, \bar{\mathbf{x}}_d'', \bar{\mathbf{x}}\big\} \subset \mathbf{A}.$$

Consequently, if $CH(\mathbf{S})$ is the convex hull of **S** then

- if $d = 2$ then the area of $CH(\{\mathbf{0}, (\bar{x}_1, 0), (\bar{x}_1, \bar{x}_2)\})$ and $CH(\{\mathbf{0}, (0, \bar{x}_2), (\bar{x}_1, \bar{x}_2)\})$ are both equal to $(\bar{x}_1 \cdot \bar{x}_2)/2!$;
- if $d > 2$ then the volume of $CH(\{\mathbf{0}, \bar{\mathbf{x}}_1', \ldots, \bar{\mathbf{x}}_k', \bar{\mathbf{x}}_{k+1}'', \ldots, \bar{\mathbf{x}}_d'', \bar{\mathbf{x}}\})$ is greater than $(1/d!) \cdot \bar{x}_1 \cdot \cdots \cdot \bar{x}_d$.

Finally, if $\bar{x}_1 \cdot \cdots \cdot \bar{x}_d \neq O(n)$, it would contradict to $\#\mathbf{A} = n$. (This follows from a theorem of Davenport [3] which estimates $|\text{volume}(\mathcal{B}) - \#(\mathcal{B} \cap \mathbb{N}^d)|$ for a closed bounded d-dimensional body $\mathcal{B}$.) This proves (2).

To prove (3), observe that

$$\big\{(\bar{x}_{i,1}, \ldots, \bar{x}_{i,d}) \colon 1 \leqslant i \leqslant d\big\} \subset \mathbf{A} \quad \Rightarrow \quad \frac{1}{d} \cdot \left(\left\lfloor \sum_{i=1}^{d} \bar{x}_{i,1} \right\rfloor, \ldots, \left\lfloor \sum_{i=1}^{d} \bar{x}_{i,d} \right\rfloor \right) \in \mathbf{A}$$

and apply (2). (Here $\lfloor x \rfloor$ is the largest integer not exceeding x.) □

We also need a well-known result (see [5]) related to the number of integer solutions of $x_1 \cdot \cdots \cdot x_n \leqslant y$.

Proposition 4. *For any given d*

$$\#\big\{\mathbf{x} \in \mathbb{N}^d \colon x_1 \cdot \cdots \cdot x_d \leqslant y\big\} = y \cdot P_d(\log y) + O\big(y^{(d-1)/d}\big)$$

holds, where $P_d(y)$ is a polynomial of degree $d-1$.

We are now ready to prove the main result of the paper.

Theorem 5. *Fix d. Then the number of n-point sphere corner cuts is*

$$O\big(n^{d+1} (\log n)^{d-1}\big).$$

Proof. The product $\mu_1(\mathbf{A}) \cdot \cdots \cdot \mu_d(\mathbf{A})$ can be expanded as a sum of n^d summands, all of which has order $O(n)$ (by Lemma 3). Thus,

$$\prod_{i=1}^{d} \mu_i(\mathbf{A}) = O\left(n^{d+1}\right). \tag{5}$$

Since $\mu_i(\mathbf{A})$, $1 \leqslant i \leqslant d$, are integers satisfying (5), then Proposition 4 gives

$$\#\left\{\left(\mu_1(\mathbf{A}), \ldots, \mu_d(\mathbf{A})\right): \ \mathbf{A} \in \binom{\mathbb{N}_0^d}{n}_{\mathrm{cut}}^{s}\right\} = O\left(n^{d+1}(\log n)^{d-1}\right).$$

The theorem now follows by Theorem 2. □

4. Cutting corners by hyperplanes

A set $\mathbf{A} \subset \mathbb{N}_0^d$ which includes $\mathbf{0}$ is called a *corner cut* if it is separable from $\mathbb{N}_0^d \setminus \mathbf{A}$ by a hyperplane. An *n-point corner cut* is a corner cut consisting of n points.

The characterization and enumeration problem related to the cutting corners by hyperplanes are considered recently [2,7,8]. Since a (hyperplane) corner cut is a sphere corner cut, the results (2), (3), and (5) hold for corner cuts as well. However, if our method is applied to corner cuts, it gives a little bit more.

Let $\mathbf{A}$, $\mathbf{B}$ be n-point corner cuts and let $\{\mathbf{x}\colon\ a_1x_1 + \cdots + a_dx_d + h_a = 0\}$ separates $\mathbf{A}$ and $\mathbb{N}_0^d \setminus \mathbf{A}$ while $\{\mathbf{x}\colon\ b_1x_1 + \cdots + b_dx_d + h_b = 0\}$ separates $\mathbf{B}$ and $\mathbb{N}_0^d \setminus \mathbf{B}$. Then

$$\left\{\mathbf{x}\colon\ (a_1b_d - a_db_1)x_1 + \cdots + (a_{d-1}b_d - a_db_{d-1})x_{d-1} + h_ab_d - h_ba_d = 0\right\}$$

separates $\mathbf{A} \setminus \mathbf{B}$ and $\mathbf{B} \setminus \mathbf{A}$. Now, as in Theorem 2, it can be shown that the mapping

$$\mathbf{A} \to \left(\mu_1(\mathbf{A}), \ldots, \mu_{d-1}(\mathbf{A})\right) \tag{6}$$

is one-to-one. Further, for the corner cuts with

$$\mu_1(\mathbf{A}) \geqslant \mu_2(\mathbf{A}) \geqslant \cdots \geqslant \mu_d(\mathbf{A}) \tag{7}$$

we have $\mu_d(\mathbf{A}) = O(n^{1+1/d})$ (because of (5)) which leads to

$$\prod_{i=1}^{d-1} \mu_i(\mathbf{A}) = O\left(n^{d-1/d}\right). \tag{8}$$

Since the number of n-point corner cuts is at most $d!$ times the number of n-point corner cuts which satisfy (7), we have the following theorem.

Theorem 6. *If d is fixed, the number of n-point corner cuts has order*

$$O\left(n^{d-1/d}(\log n)^{d-1}\right).$$

5. Concluding remarks

A variety of problems similar to those considered here comes from the area of digital geometry and digital image analysis where real objects are usually replaced by finite point subsets of $\mathbb{N}_0^2$ or $\mathbb{N}_0^3$ (see [1,4,6,9,10]).

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